MATH 301 **TEST 2** Solution Key
name

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<u>For full credit show all your work. Good luck!</u>

1. (10 pts) The function $f(x, y) = \frac{x^3 y}{x^4+y^4}$ is undefined at (0, 0). Does it have a **limit** as (x, y) approaches (0, 0). ? Justify your answer.

$$\lim_{(x,y)\to(0,0)} \frac{x^3y}{x^4+y^4} = \lim_{x\to 0} \frac{x^3x}{x^4+x^4} = \lim_{x\to 0}\frac{x^4}{2x^4} = \frac{1}{2}$$

along the line $y=x$

$$\lim_{(x,y)\to(0,0)} \frac{x^3y}{x^4+y^4} = \lim_{x\to 0} \frac{0}{x^4+0} = 0$$

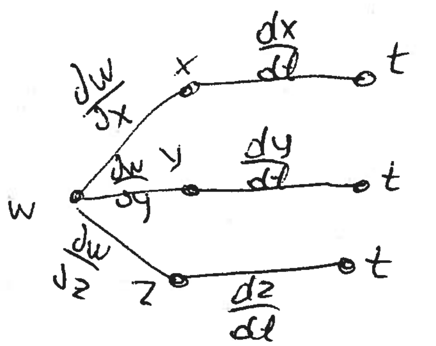
along the x-axis ($y=0$)

Since the limits along two approach paths are different, $f(x,y)$ has no limit as $(x,y)\to(0,0)$.

2. (15 pts) Draw a **dependence diagram** for the composition of functions

w = f(x, y, z), x = g(t), y = h(t), z = k(t)

Use this diagram **to find a formula** for dw/dt (be careful in using "d" and "del")



$$\frac{dw}{dt} = \frac{\partial w}{\partial x}\cdot\frac{dx}{dt} + \frac{\partial w}{\partial y}\cdot\frac{dy}{dt} + \frac{\partial w}{\partial z}\cdot\frac{dz}{dt}$$

or

$$f'(t) = f_x\cdot g'(t) + f_y\cdot h'(t) + f_z\cdot k'(t)$$

Find dw/dt if w = xy + xz + yz, x = cos t, y = sin 2t, and z = 3t .

$$\frac{dw}{dt} = (y+z)(-\sin t) + (x+z)2\cos 2t + (x+y)\cdot 3$$
$$= (\sin 2t + 3t)(-\sin t) + (3t+\cos t)2\cos 2t + 3(\cos t + \sin 2t)$$

3. (25 pts) A particle moves along the space curve $\bar{r}(t) = 2(1+t)^{3/2}\,\mathbf{i} + 2(1-t)^{3/2}\,\mathbf{j} + 3t\,\mathbf{k}$.

(a) Find the **velocity**, **acceleration**, **speed**, and **unit tangent vector** for this motion.

velocity $= \bar{v}(t) = \bar{r}'(t) = 3(1+t)^{\frac{1}{2}}\,\bar{i} - 3(1-t)^{1/2}\,\bar{j} + 3\bar{k}$

acceleration $= \bar{a}(t) = \bar{r}''(t) = \frac{3}{2}(1+t)^{-\frac{1}{2}}\,\bar{i} + \frac{3}{2}(1-t)^{-\frac{1}{2}}\,\bar{j}$

speed $= |\bar{v}(t)| = \sqrt{9(1+t)+9(1-t)+9} = \sqrt{27} = 3\sqrt{3}$

9 unit tangent

$$\bar{T}(t) = \frac{\bar{v}(t)}{|\bar{v}(t)|} = \left\langle \sqrt{\frac{1+t}{3}},\ -\sqrt{\frac{1-t}{3}},\ \frac{1}{\sqrt{3}} \right\rangle$$

(b) Find the **curvature** of the particle's path for t = 0 (do not work with general t, use t=0). $\bar{V}(0) = \langle 3, -3, 3 \rangle$, $\bar{a}(0) = \langle \frac{3}{2}, \frac{3}{2}, 0 \rangle$

8 $$\bar{v}(0) \times \bar{a}(0) = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ 3 & -3 & 3 \\ 3/2 & 3/2 & 0 \end{vmatrix} = \left\langle -\frac{9}{2}, \frac{9}{2}, 9 \right\rangle$$

$$K(0) = \frac{|\langle -\frac{9}{2}, +\frac{9}{2}, 9 \rangle|}{|\langle 3, -3, 3 \rangle|^3} = \frac{\sqrt{\frac{81}{4}+\frac{81}{4}+81}}{(3\sqrt{3})^3} = \frac{9 \cdot \sqrt{\frac{3}{2}}}{9 \cdot 3 \cdot 3\sqrt{3}} = \boxed{\frac{\sqrt{2}}{18}}$$

(c) Find **the normal and tangential components of acceleration** for this motion at t = 0.

$$a_T(0) = \frac{\bar{v} \cdot \bar{a}}{|\bar{v}|} = \frac{0}{|\bar{v}|} = \boxed{0}$$

4 $$a_N(0) = \frac{|\bar{v} \times \bar{a}|}{|\bar{v}|} = \frac{9 \cdot \sqrt{\frac{3}{2}}}{3\sqrt{3}} = \frac{3}{\sqrt{2}} = \boxed{\frac{3\sqrt{2}}{2}}$$

(d) What is the **distance traveled by the particle** between the points (2, 2, 0) and $(4\sqrt{2}, 0, 3)$?

$\langle 2, 2, 0 \rangle = \bar{r}(0)$, $\langle 4\sqrt{2}, 0, 3 \rangle = \bar{r}(1)$

4 $$L = \int_0^1 |\bar{v}(t)|\,dt = \int_0^1 3\sqrt{3}\,dt = 3\sqrt{3}\,t\Big|_0^1 = \boxed{3\sqrt{3}}$$

4. (10 pts) Find an equation of the **plane tangent** to the surface $x^3 + 2xy + y^3 + 2yz + z^3 = 10$ at the point (-1, 1, 2).

The surface is a level curve of $f(x,y,z) = x^3 + 2xy + y^3 + 2yz +$

for $c = 10$.

$f_x = 3x^2 + 2y$ $\quad$ $f_y = 2x + 3y^2 + 2z$ $\quad$ $f_z = 2y + 3z^2$

$f_x(-1,1,2) = 5$ $\quad$ $f_y(-1,1,2) = 5$ $\quad$ $f_z(-1,1,2) = 14$

Equation of the tangent plane:

$5(x+1) + 5(y-1) + 14(z-2) = 0$ or $\boxed{5x + 5y + 14z = 28}$

5. (15 pts) Let $f(x, y) = 2x^2 + x\cos(xy)$

(a) Find **the gradient** of f at P(1, 0) and the **directional derivative** of f at P in the direction of **a** = - 3**i** + 4**j** .

$$\bar{u} = \frac{\bar{a}}{|\bar{a}|} = \frac{\langle -3, 4\rangle}{5} = \langle -\tfrac{3}{5}, \tfrac{4}{5}\rangle$$

$f_x = 4x + \cos(xy) - xy\sin(x,y)$, $\quad f_x(1,0) = 5$

$f_y = -x^2\sin(xy)$ $\quad f_y(1,0) = 0$

$\nabla f(1,0) = \langle 5, 0\rangle$, $\quad D_{\bar{u}} f(1,0) = \langle 5, 0\rangle \cdot \langle -\tfrac{3}{5}, \tfrac{4}{5}\rangle = -3$

(b) Find the **unit vector** in the direction of the most rapid increase of f at P and the **rate of increase** of f in that direction.

f increases most rapidlly in the direction of $\langle 5, 0\rangle$ or $\bar{u} = \bar{i}$.

Rate of increase $= |\langle 5, 0\rangle| = 5$.

6. (15 pts) The length of the hypotenuse of the right triangle with legs of length x an y is $h(x, y) = (x^2 + y^2)^{1/2}$.

(a) Considering h as a function of two variables x and y find its **total differential** dh

$$dh = h_x\,dx + h_y\,dy = \frac{x}{\sqrt{x^2+y^2}}\,dx + \frac{y}{\sqrt{x^2+y^2}}\,dy$$ 6pts

(b) Find **the length of the hypotenuse of the triangle** with legs x = 5 and h = 12 (measured in inches)

$$h(5,12) = \sqrt{5^2+12^2} = \sqrt{25+144} = 13$$ 3pts

(c) Use the differential from part (a) to **estimate the change in the length of the hypotenuse** of the triangle if x increases from 5 to 5.1 inches and y decreases from 12 to 11.95 inches.

Here $dx = 0.1$ and $dy = -0.05$

$$\Delta h \approx dh = \frac{5}{13}\,0.1 + \frac{12}{13}\cdot(-0.05) = \frac{0.5-0.6}{13} = \boxed{-\frac{1}{130}}$$ 6pts

7. (15 pts) Find the **domain** of the function $f(x, y) = \ln(x + y)$.
Sketch (and label) the **level curves** $f(x, y) = c$ for $c = 0, c = -1$ and $c = 1$.
Evaluate the **gradient** of f(x, y) at the point P(2, -1) and sketch it (with the initial point at P).

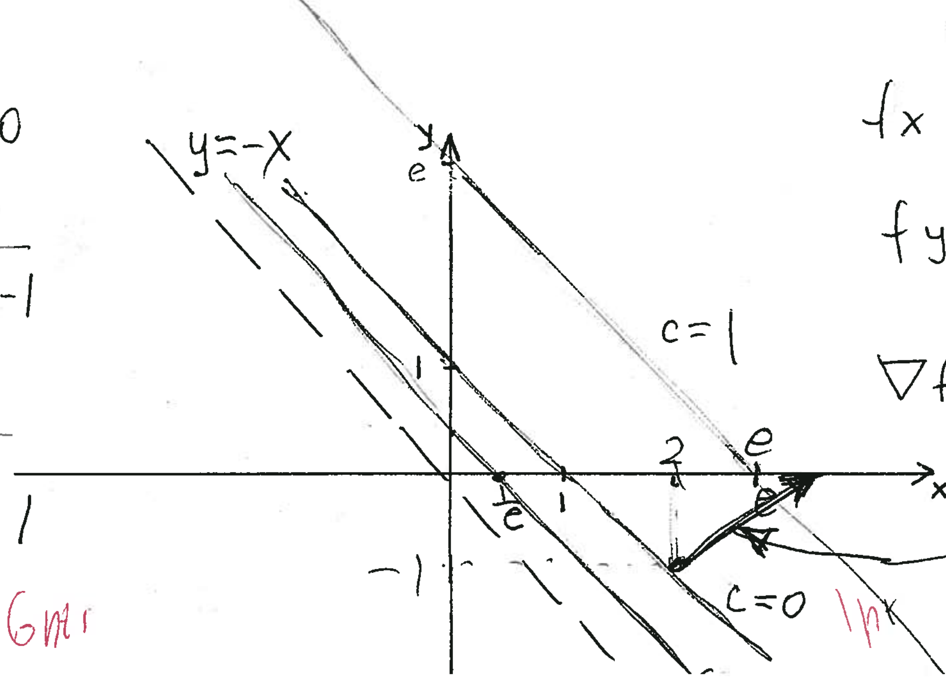
3pts $\mathcal{D}_f = \{(x,y) \text{ in } R^2 : x+y>0\}$ (all points above the line $y=-x$)

Level curves:

$c=0$ $\ln(x+y)=0$
$x+y=1$

$c=-1$ $\ln(x+y)=-1$
$x+y=\frac{1}{e}$

$c=1$ $\ln(x+y)=1$
$x+y=e$ 6pts



$f_x = \frac{1}{x+y}$

$f_y = \frac{1}{x+y}$ 4pts

$\nabla f(2,-1) = \langle 1, 1\rangle$ 1pt