MATH 301 **Test 1** Practice

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Solution Key
(name)

For full credit show all your work. Good luck !

1. (15 pts) (a) Find a vector **v** that is parallel to the line l given by:

$x = 1 - t$
$y = 2 + 3t$
$z = 2t$

$\bar{v} = \langle -1, 3, 2 \rangle$

(or any multiple of it).

(b) Let P(0, 2, 5) be a point outside the line l. Find the **distance** between the point P and the line l.

Q(1, 2, 0) is a point on the line

$\overline{QP} = \langle -1, 0, 5 \rangle$

$\overline{QP} \times \bar{v} = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ -1 & 0 & 5 \\ -1 & 3 & 2 \end{vmatrix} = -15\bar{i} - 3\bar{j} - 3\bar{k}$

$d = \dfrac{|\overline{QP} \times \bar{v}|}{|\bar{v}|} = \dfrac{\sqrt{225+9+9}}{\sqrt{1+9+4}}$

$d = \sqrt{\dfrac{243}{14}}$

(c) Find an **equation of the plane** containing both the point P and the line l.

$\bar{n} = \overline{QP} \times \bar{v} = -15\bar{i} - 3\bar{j} - 3\bar{k}$ is a vector normal to the plane. Its equation is

$-15(x-0) - 3(y-2) - 3(z-5) = 0$ or $\boxed{5x + y + z = 7}$

2. (10 pts) Find an **equation** of:

(a) The sphere centered at A(-2, 3, -6) and passing through the origin.

radius $= d(A, O(0,0,0)) = \sqrt{(-2)^2 + 3^2 + (6)^2} = 7$

Equation: $\boxed{(x+2)^2 + (y-3)^2 + (z+6)^2 = 7^2}$

(b) The circular cylinder around the z-axis and passing through the point B(1, 2, 2)

radius of the cylinder $= d(B(1,2,2), Z(0,0,2)) = \sqrt{5}$

Equation: $x^2 + y^2 = (\sqrt{5})^2$ or $\boxed{x^2 + y^2 = 5}$

3. (10 pts) Find **parametric equations** of the line containing the point (1, 1, 5) and perpendicular to the plane $x - 3y + 4z = 12$.

The line is parallel to the normal vector

$\bar{n} = \langle 1, -3, 4 \rangle$

Parametric equations:

$$\boxed{\begin{aligned} x &= 1 + t \\ y &= 1 - 3t \\ z &= 5 + 4t \end{aligned}}$$

4. (15 pts) Two lines in space are given by the parametric equations:
L_1: $x = 2 + t$, $y = 1 + t$, $z = -1 + t$
L_2: $x = 2 + 3t$, $y = 1$, $z = -1 - t$

(a) Justify that these two lines **have a point in common.**

Point $P(2, 1, -1)$ lies on L_1 (for $t = 0$) and on L_2 (also for $t = 0$)

(b) Find an equation of the **plane** containing both lines L_1 and L_2.

a normal vector of the plane is perpendicular to both lines: $\bar{n} = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ 1 & 1 & 1 \\ 3 & 0 & -1 \end{vmatrix} = -\bar{i} + 4\bar{j} - 3\bar{k}$

$-(x-2) + 4(y-1) - 3(z+1) = 0$ or $\boxed{-x + 4y - 3z = 5}$

5. (5 pts) Does there exist a vector **v** such that $\mathbf{v} \times \mathbf{v} = 4\mathbf{k}$? Justify your answer.

No, since $\bar{v} \times \bar{v} = \bar{0}$ for any vector $\bar{v}$.

6. (15 p) Two planes M_1 and M_2 in space are given by the equations:

$M_1: \; 3x + y - 2z = 0$
$M_2: \; x + 2z = 5$

(a) Find the **normal vector** of each of the two planes.

$\bar{n}_1 = \langle 3, 1, -2 \rangle$

$\bar{n}_2 = \langle 1, 0, 2 \rangle$

(b) Justify that the point P(1, 1, 2) is **common to both planes**.

P satisfies both equations:

$3 \cdot 1 + 1 - 2 \cdot 2 = 0$

$1 + 2 \cdot 2 = 5$

(c) Find the **direction** of the line of intersection of the two planes and its **equation**.

direction is given by $\bar{v} = \bar{n}_1 \times n_2 = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ 3 & 1 & -2 \\ 1 & 0 & 2 \end{vmatrix}$

$= \boxed{2\bar{i} - 8\bar{j} - \bar{k}}$

Equations: $\boxed{\begin{aligned} x &= 1 + 2t \\ y &= 1 - 8t \\ z &= 2 - t \end{aligned}}$

7. (10 pts) In space, the vectors **u** and **v** are orthogonal and the vectors **v** and **w** are orthogonal. Can we claim that the vectors **u** and **w** are parallel ?
Justify your answer or give an example if you believe that the fact is not true.

Let $\bar{u} = \bar{i}$, $\bar{v} = \bar{j}$, $\bar{w} = \bar{k}$.

Then $\bar{u} \perp \bar{v}$ and $\bar{v} \perp \bar{w}$, but $\bar{u}$ and $\bar{w}$ are not parallel (in fact, in this case $\bar{u} \perp \bar{w}$).

We cannot claim that $\bar{u}$ and $\bar{v}$ are parallel.

8. (10 pts) The line $x = y = z + 2$ is parallel to the plane $x + y - 2z = 5$.
Find the distance between the line and the plane.
Hint: Select a point on the line. Find its distance from the plane.

$P(2,2,0)$ is a point on the line

$P_0(5,0,0)$ is a point on the plane

$$\bar{b} = \overline{P_0P} = \langle -3, 2, 0 \rangle$$

$$d = \frac{|\bar{n} \cdot \bar{b}|}{|\bar{n}|} = \frac{|\langle 1,1,-2\rangle \cdot \langle -3,2,0\rangle|}{|\langle 1,1,-2\rangle|} = \frac{|-3+2|}{\sqrt{1+1+4}} = \frac{1}{\sqrt{6}} = \frac{\sqrt{6}}{6}$$

9. (15 pts) A vector function is given by $\mathbf{r}(t) = 3\cos t\,\mathbf{i} + 5\sin t\,\mathbf{j} - 4\cos t\,\mathbf{k}$.

(a) Find the limit $\lim_{t \to \frac{\pi}{3}} \mathbf{r}(t)$.

$$\lim_{t \to \frac{\pi}{3}} \bar{r}(t) = \left(\lim_{t \to \frac{\pi}{3}} 3\cos t\right)\bar{i} + \left(\lim_{t \to \frac{\pi}{3}} 5\sin t\right)\bar{j} + \lim_{t \to \frac{\pi}{3}}(-4\cos t)$$

$$= 3 \cdot \frac{1}{2}\,\bar{i} + 5 \cdot \frac{\sqrt{3}}{2}\,\bar{j} - 4 \cdot \frac{1}{2}\,\bar{k} = \boxed{\frac{3}{2}\bar{i} + \frac{5\sqrt{3}}{2}\bar{j} - 2}$$

(b) Find $\mathbf{r}'(t)$ and $\mathbf{r}''(t)$.

$$\bar{r}'(t) = -3\sin t\,\bar{i} + 5\cos t\,\bar{j} + 4\sin t\,\bar{k}$$

$$\bar{r}''(t) = -3\cos t\,\bar{i} - 5\sin t\,\bar{j} + 4\cos t\,\bar{k}$$

(c) Find the unit tangent vector $\mathbf{T}(0)$.

$$\bar{T}(0) = \frac{\bar{r}'(0)}{|\bar{r}'(0)|}$$

$$\bar{r}'(0) = 0\,\bar{i} + 5\,\bar{j} + 0\,\bar{k} = 5\,\bar{j}$$

$$|\bar{r}'(0)| = 5$$

$$\bar{T}(0) = \frac{5\bar{j}}{5} = \bar{j}$$