Math 301 Test 3 Name: Solution Key.

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For full credit show all your work.

1. (15 pts) Use the Lagrange multipliers method to find the point P on the surface $xy^2 = 54$ that is closest to the origin O.
Hint: If P(x, y), then minimize $f(x, y) = x^2 + y^2$ subject to the constraint $xy^2 - 54 = 0$.

$g(x,y) = xy^2 - 54$ $\nabla g = \langle y^2, 2xy \rangle$

$\nabla f = \langle 2x, 2y \rangle$

System:

$$\begin{cases} xy^2 = 54 & (1) \\ 2x = \lambda y^2 & (2) \\ 2y = \lambda 2xy & (3) \end{cases}$$

If $x = 0$ or $y = 0$, then (1) is false.
So $x \neq 0$, $y \neq 0$ and then

from (3) $\lambda = \frac{1}{x}$, so (2) gives $2x = \frac{1}{x} y^2$ or $2x^2 = y^2$.
Then (1) becomes $2x^3 = 54$, so $x^3 = 27$, $x = 3$.
Since $y^2 = 2x^2$, $y^2 = 18$, $y = \pm 3\sqrt{2}$.
Critical points: $(3, 3\sqrt{2}), (3, -3\sqrt{2})$; There are two such P's closest to the origin because the surface $xy^2 = 54$ is unbounded.

2. (10 pts) Find the volume of the solid over the rectangle $0 \le x \le 2, 0 \le y \le 3$ and below the surface $z = xy^2$.

$$\text{Volume} = \int_0^3 \int_0^2 xy^2 \, dx \, dy = \int_0^3 \frac{1}{2} x^2 y^2 \Big|_{x=0}^{x=2} dy$$

$$= \int_0^3 2y^2 \, dy = \frac{2}{3} y^3 \Big|_0^3 = \frac{2}{3} \cdot 3^3 = \boxed{18}$$

3. (20 pts) Let $f(x, y) = x^3 + y^3 + 3x^2 - 3y^2$.
(a) Find all critical points of f.

$f_x = 3x^2 + 6x$ Solve $\begin{cases} 3x^2+6x=0 \\ 3y^2-6y=0 \end{cases}$

4pts $f_y = 3y^2 - 6y$

4pts $\begin{cases} 3x(x+2)=0 \\ 3y(y-2)=0 \end{cases}$ $x=0$ or $x=-2$, $y=0$ or $y=2$

Critical points $(0,0), (0,2)$, $(-2,0), (-2,2)$

(b) Use the second partials test to determine at which points f has local extrema and at which it has saddle points.

4pts $D(x,y) = \begin{vmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{vmatrix} = \begin{vmatrix} 6x+6 & 0 \\ 0 & 6y-6 \end{vmatrix} = 36(x+1)(y-1)$

8pts

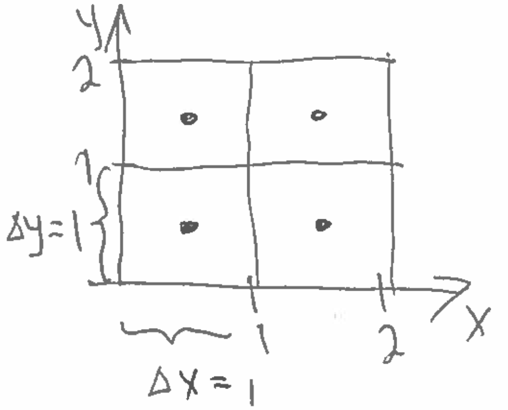
$D(0,0) = -36 \Rightarrow (0,0)$ is a saddle point

$D(0,2) = 36$ and $f_{xx}(0,2) = 6 \Rightarrow$ local minimum $f(0,2) = -4$

$D(-2,0) = 36$ and $f_{xx}(-2,0) = -6 \Rightarrow$ local maximum $f(-2,0) = 4$

$D(-2,2) = -36 \Rightarrow (-2,2)$ is a saddle point

4. (10 pts) Consider the function $f(x, y) = x^2 + 2y$ defined on the square R: $0 \le x \le 2, 0 \le y \le 2$. Find the Riemann sum for f if R is subdivided into four 1 by 1 squares and the chosen points from each of the small squares are their midpoints.



Here $\Delta x = 1$, $\Delta y = 1$, so

Riemann sum =

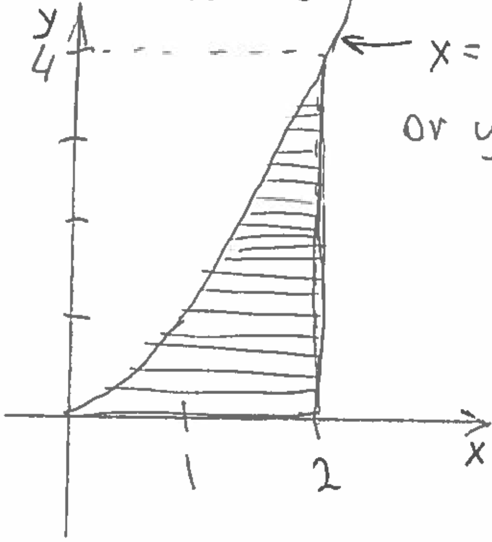
$= \left[f\left(\frac{1}{2}, \frac{1}{2}\right) + f\left(\frac{3}{2}, \frac{1}{2}\right) + f\left(\frac{1}{2}, \frac{3}{2}\right) + f\left(\frac{3}{2}, \frac{3}{2}\right)\right] \Delta x \Delta y$

$= \left(\frac{1}{4}+1\right) + \left(\frac{9}{4}+1\right) + \left(\frac{1}{4}+3\right) + \left(\frac{9}{4}+3\right) = \boxed{13}$

5. (15 pts) The iterated double integral

$$\int_0^4 \int_{\sqrt{y}}^2 (x^2+y^2)\,dx\,dy \;=\; I$$

is over Type II region D. Draw the region D in the plane and rewrite the integral changing the order of integration producing the integral over Type I region.



$x = \sqrt{y}$
or $y = x^2$

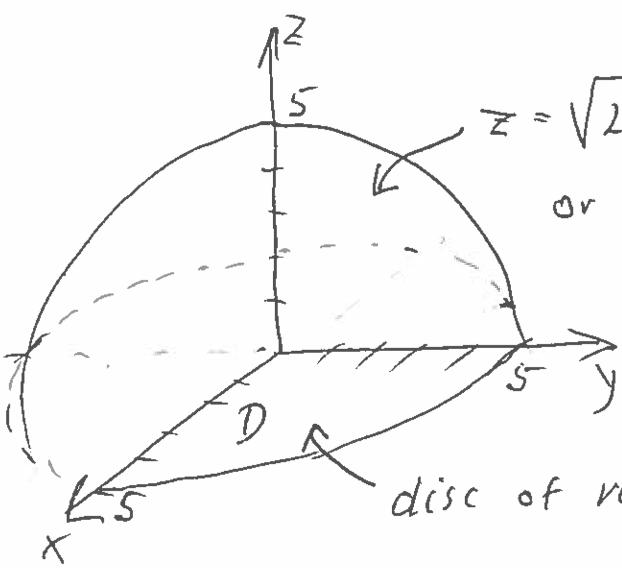
$D: \; 0 \le y \le 4$
$\sqrt{y} \le x \le 2$

As type I region:

$D: \; 0 \le x \le 2$
$0 \le y \le x^2$

$$I = \int_0^2 \int_0^{x^2} (x^2+y^2)\,dy\,dx$$

6. (10 pts) Express the volume of the solid above the xy-plane and below the upper hemisphere $z = \sqrt{25 - x^2 - y^2}$ by a double integral in polar form. Do not evaluate it.



$z = \sqrt{25 - x^2 - y^2}$
or $z = \sqrt{25 - r^2}$

disc of radius 5

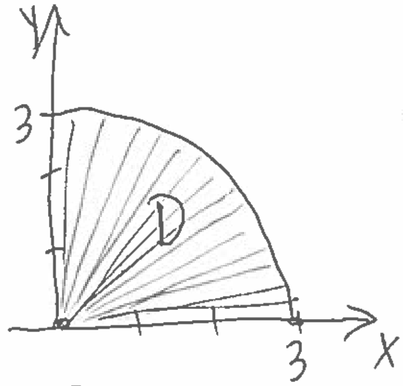
$D: \; 0 \le \theta \le 2\pi$
$0 \le r \le 5$

$$\boxed{\text{Volume} = \int_0^{2\pi}\int_0^5 \sqrt{25 - r^2}\; r\,dr\,d\theta}$$

7. (13 pts) Use polar coordinates to evaluate the double integral

$$\iint_D \cos(x^2+y^2+1)\,dA,$$

if D is the portion of the disc of radius 3 in the first quadrant.



$D: \quad 0 \le \theta \le \frac{\pi}{2}$

$\quad\;\; 0 \le r \le 3$

$$\iint_D \cos(x^2+y^2+1)\,dA =$$

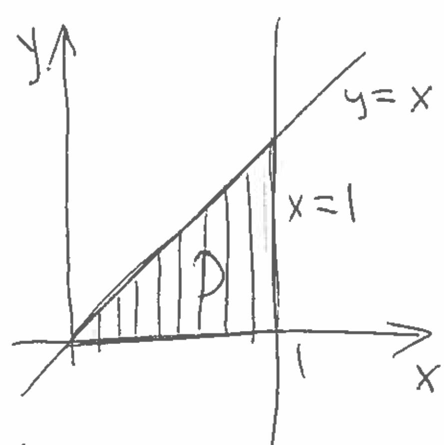
$$= \int_0^{\frac{\pi}{2}}\int_0^3 \cos(r^2+1)\, r\,dr\,d\theta = \int_0^{\frac{\pi}{2}} \frac{1}{2}\sin(r^2+1)\Big|_0^3 d\theta = \frac{\pi}{2}\cdot\frac{1}{2}(\sin 10 - \sin 1)$$

$$= \boxed{\frac{\sin 10 - \sin 1}{4}\cdot \pi}$$

8. (12 pts) Evaluate the double integral

$$\iint_D e^{x+y}\,dA,$$

if D is the triangle in the first quadrant bounded by the lines $y = x$, $x = 1$, and the x-axis.



$D: \quad 0 \le x \le 1$

$\quad\;\; 0 \le y \le x$

$$\iint_D e^{x+y}\,dA = \int_0^1\int_0^x e^{x+y}\,dy\,dx$$

$$= \int_0^1 e^{x+y}\Big|_{y=0}^{y=x} dx = \int_0^1 (e^{2x} - e^x)\,dx = \left(\frac{1}{2}e^{2x} - e^x\right)\Big|_0^1 =$$

$$= \frac{e^2}{2} - e - \frac{1}{2} + 1 = \boxed{\frac{e^2}{2} - e + \frac{1}{2}}$$