MATH 301 **Test 4** Solution Key

Spring 2017 Name

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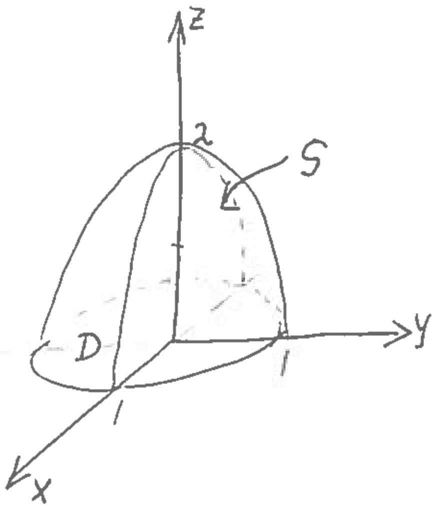
For full credit show all your work. Good luck!

1. (15 pts) Surface S is the portion of the ellipsoid $4x^2 + 4y^2 + z^2 = 4$ lying above the xy-plane. Express the **area** of S as a double integral over a suitable region D in the xy-plane. **Sketch** S and D. Do **not** evaluate the double integral.



For $z=0$: $4x^2+4y^2=4$

$x^2+y^2=1$

so D is a disc of radius 1

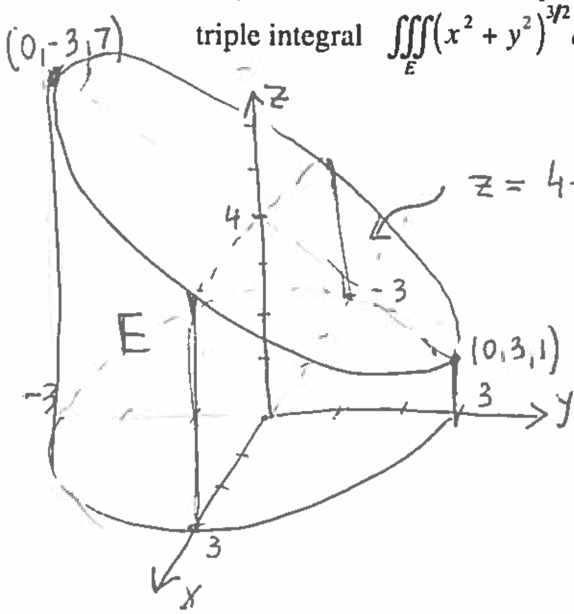
Surface S:

$$z=f(x,y)=\sqrt{4-4x^2-4y^2}=2\sqrt{1-x^2-y^2}$$

$$f_x=\frac{-2x}{\sqrt{1-x^2-y^2}}, \quad f_y=\frac{-2y}{\sqrt{1-x^2-y^2}}$$

$$\text{Area}(S)=\iint_D \sqrt{f_x^2+f_y^2+1}\,dA=\int_{-1}^{1}\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}}\sqrt{\frac{4x^2+4y^2}{1-x^2-y^2}+1}\,dy\,dx$$

2. (10 pts) Let E be the solid region bounded above by the plane $y + z = 4$, below by the xy plane, and on the sides by the cylinder $x^2 + y^2 = 9$. **Sketch** the region E. Express the triple integral $\iiint_E (x^2+y^2)^{3/2}\,dV$ in **cylindrical coordinates**. Do **not** evaluate it.



$z=4-y=4-r\sin\theta$

E: $0\le\theta\le 2\pi$

$0\le r\le 3$

$0\le z\le 4-r\sin\theta$

$$\iiint_E (x^2+y^2)^{3/2}\,dV=\int_0^{2\pi}\int_0^3\int_0^{4-r\sin\theta} r^3\cdot r\,dz\,dr\,d\theta$$

3. (12 pts) Use the **spherical coordinates** to find the **mass** of a ball of radius 5 centered at the origin and having the mass density $g(x,y,z) = 1 / (x^2 + y^2 + z^2)$.

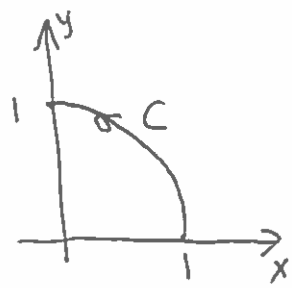
$$E: \quad 0 \le \theta \le 2\pi, \quad 0 \le \phi \le \pi, \quad 0 \le \rho \le 5$$

$$g(r,\theta,\phi) = \frac{1}{\rho^2}$$

$$\text{Mass} = \iiint_E g(x,y,z)\,dV = \int_0^{2\pi}\int_0^{\pi}\int_0^{5} \frac{1}{\rho^2}\cdot\rho^2 \sin\phi \, d\rho\, d\phi\, d\theta$$

$$= (2\pi)\left(-\cos\phi\right)\Big|_{\phi=0}^{\phi=\pi}\cdot 5 = (2\pi)(2)\,5 = \boxed{20\pi}$$

4. (13 pts) Use an unoriented line integral to evaluate the **mass of the wire** in the shape of a unit circle (in the first quadrant) with the mass density $f(x, y) = xy$.



$$\bar{r}(t) = \cos t\,\bar{i} + \sin t\,\bar{j}, \quad 0 \le t \le \frac{\pi}{2}$$

$$\bar{r}'(t) = -\sin t\,\bar{i} + \cos t\,\bar{j}$$

$$|\bar{r}'(t)| = \sqrt{\sin^2 t + \cos^2 t} = 1$$

$$\text{Mass} = \int_C xy\,ds = \int_0^{\frac{\pi}{2}} \cos t \sin t \cdot 1 \cdot dt = \int_0^{\frac{\pi}{2}} \frac{1}{2}\sin 2t\,dt$$

$$= -\frac{1}{4}\cos 2t\,\Big|_0^{\frac{\pi}{2}} = \frac{1}{4} - \left(-\frac{1}{4}\right) = \boxed{\frac{1}{2}}$$

5. (15 pts) The region D in the xy-plane is bounded by the lines x = 2, x = 6, and by the hyperbolas xy = 1, xy = 4. **Evaluate the area of** D.

Hint: Use the following change of variables in a double integral: x = u, y = v/u.

Boundaries:

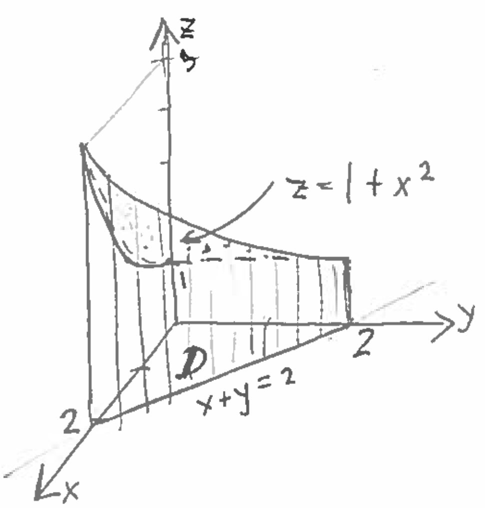
in xy-coords	$x=2$	$x=6$	$xy=1$	$xy=4$
in uv-coords	$u=2$	$u=6$	$u\cdot\frac{v}{u}=1$ or $v=1$	$u\cdot\frac{v}{u}=4$, $v=4$

Jacobian: $\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} 1 & 0 \\ -\frac{v}{u^2} & \frac{1}{u} \end{vmatrix} = \frac{1}{u}$

$$\text{Area}(D) = \iint_D 1\, dA = \int_2^6 \int_1^4 \left|\frac{1}{u}\right| dv\, du = \int_2^6 \frac{1}{u} v \Big|_{v=1}^{v=4} du =$$

$$= 3\int_2^6 \frac{1}{v}\, dv = 3 \ln|v| \Big|_2^6 = 3(\ln 6 - \ln 2) = \boxed{3\ln 3}$$

6. (15 pts) Find the **volume** of the solid lying in the first quadrant and bounded by the coordinate planes, the plane x + y = 2, and the parabolic sheet $z = 1 + x^2$.



$D: \; 0 \le x \le 2$
$\quad\;\; 0 \le y \le 2-x$

solid E: $0 \le x \le 2$
$\quad\quad\quad 0 \le y \le 2-x$
$\quad\quad\quad 0 \le z \le 1+x^2$

$$\text{Volume} = \iiint_E 1\, dV = \int_0^2 \int_0^{2-x} \int_0^{1+x^2} dz\, dy\, dx = \int_0^2 \int_0^{2-x} (1+x^2)\, dy\, dx$$

$$= \int_0^2 (1+x^2)(2-x)\, dx = \int_0^2 (2 - x + 2x^2 - x^3)\, dx = \left(2x - \frac{1}{2}x^2 + \frac{2}{3}x^3 - \frac{1}{4}x^4\right)\Big|_0^2$$

$$= 4 - 2 + \frac{16}{3} - 4 = \boxed{\frac{10}{3}}$$

7. (15 pts) (a) Justify that the vector field $\mathbf{F}(x, y) = (6x^2y + x + 4y)\,\mathbf{i} + (2x^3 + 4x - 3)\mathbf{j}$ is **conservative.**

$\frac{\partial P}{\partial y} = 6x^2+4$, $\frac{\partial Q}{\partial x} = 6x^2+4$. $\bar{F}$ is conservative since $\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$

(b) Find its **potential** $f(x, y)$.

$$\frac{\partial f}{\partial x} = 6x^2y + x + 4y \Rightarrow f(x,y) = 2x^3y + \frac{1}{2}x^2 + 4xy + \varphi(y)$$

$$\frac{\partial f}{\partial y} = 2x^3 + 4x + \varphi'(y) = 2x^3 + 4x - 3$$

so $\varphi'(y) = -3$

$\varphi(y) = -3y + C$

Therefore, $f(x,y) = 2x^3y + \frac{1}{2}x^2 + 4xy - 3y + C$

(c) Use the **Fundamental Theorem of Line Integrals** to evaluate $\int_C \vec{F} \cdot d\vec{r}$. where C is any smooth curve in the plane between the points $(0, 0)$ and $(1, 1)$.

$$\int_C \vec{F} \cdot d\bar{r} = f(1,1) - f(0,0) = \left(2 + \frac{1}{2} + 4 - 3\right) - 0 = \boxed{\frac{7}{2}}$$

8. (10 pts) Find the **work** done by the force $\mathbf{F} = 2x\,\mathbf{i} + y\,\mathbf{j} + (z - x)\,\mathbf{k}$ in moving an object along the linear segment path from $(1, 1, 1)$ to $(2, 3, 4)$.

vector equation of the line segment C:

$$\bar{r}(t) = (1+t)\bar{i} + (1+2t)\bar{j} + (1+3t)\bar{k}, \qquad 0 \le t \le 1$$

$$\bar{r}'(t) = \bar{i} + 2\bar{j} + 3\bar{k}$$

$$\text{Work} = W = \int_C \bar{F} \cdot d\bar{r} = \int_0^1 \bar{F}(\bar{r}(t)) \cdot \bar{r}'(t)\,dt =$$

$$= \int_0^1 \langle 2(1+t),\ 1+2t,\ 1+3t-1-t \rangle \cdot \langle 1, 2, 3 \rangle\,dt$$

$$= \int_0^1 (2+2t+2+4t+6t)\,dt = \int_0^1 (4+12t)\,dt = (4t+6t^2)\Big|_0^1$$

$$= 4 + 6 = \boxed{10}.$$