Mathematics 301 - Sample Final

1. For the planes P_1: $x - 3y + 4z = 3$ and P_2: $-x + 2y = 2$ find:
   a) a normal vector for each of the two planes.
   b) a point (x, y, z) common to both planes.
   c) an equation for the line common to both planes.

2. For a particle moving along the space curve $\mathbf{r} = t\mathbf{i} - 2\cos t\mathbf{j} - 2\sin t\mathbf{k}$:
   a) find the velocity $\mathbf{v}(t)$
   b) find the acceleration $\mathbf{a}(t)$
   c) find the speed at $t = \dfrac{\pi}{2}$
   d) Find the length of the curve between $t = 0$ and $t = \pi$
   e) Find the curvature $\kappa(t)$

3. Find an equation of the plane tangent to the surface $z = 4x^2 - y^2$ at the point $(5, -8, 36)$.

4. For the function $f(x,y) = 3x^2 - xy$:
   a) Find the differential df of $f(x,y)$.
   b) Approximate $f(1.01, 1.98)$ using the differential df.

5. For the function $f(x,y,z) = \sqrt{x^2 + y^2 + z^2}$:
   a) Compute the directional derivative of f at $(-2, 2, 1)$ in the direction of $\mathbf{a} = \mathbf{i} - 2\mathbf{k}$.
   b) Find the direction in which f increases most rapidly at the point $(-2, 2, 1)$ and give the numerical rate of increase in f in that direction.

6. Either evaluate $\lim\limits_{(x,y)\to(0,0)} f(x,y)$ or show that it does not exist for the function

$$f(x) = \begin{cases} \left(x + \dfrac{2}{x}\right)y & \text{if } x \neq 0 \\ 0 & \text{otherwise} \end{cases}$$

7. Use the Chain Rule to find a formula for $\dfrac{\partial z}{\partial x}$ where: $z = r^3 + sv^2$, $r = xe^y$, $s = ye^x$, $v = x^2y$.

8. Find the point on the plane $2x + y + 4z = 12$ for which $f(x,y,z) = 4x^2 + y^2 + 5z^2$ has its minimum value.

9. For the function $f(x,y) = x^3 - 2x^2 - 2xy + y^2$:
   a) Find all critical points of $f(x,y)$
   b) Using the second derivative test, determine where f has extrema and where it has saddle points. Evaluate f at each critical point and classify which type of extreme value its graph has there.

10. For the double integral $I = \int_0^2 \int_{2y}^4 e^{-x^2}\,dx\,dy$
    a) Sketch the region of integration R.
    b) Reverse the order of integration and evaluate the resulting integral.

11. Compute $\iint_R x^3 y\,dA$ where R is the region in the first quadrant bounded by $y = x^3$ and $y = x^2$.

12. Find an integral (<u>but do not integrate</u>) that represents the surface area of the sphere $x^2 + y^2 + z^2 = 4$ above the upper nappe of the cone $x^2 + y^2 = z^2$.

13. Find the volume of the solid that is bounded above by the plane $y + z = 1$, below by the xy-plane and on the sides by the parabolic sheet $y = x^2$.

14. Find a formula in cylindrical coordinates for the mass of an object occupying the region common to both the cylinder $x^2 + y^2 = 4$ and the sphere $x^2 + y^2 + z^2 = 9$ above the xy-plane if the mass density function is equal to the distance from the z-axis. <u>Do not evaluate</u>.

15. Find the volume of the spherical shell with inner radius 2 and outer radius 3 lying above the upper nappe of the cone $3x^2 + 3y^2 = z^2$.

16. Show that $\int_C (2xy - 2y^2)dx + (x^2 - 4xy - 4)dy$ is independent of path and evaluate it. Assume that C is a piecewise smooth curve from (0, 1) to (–2, 3).

17. Use Green's Theorem to evaluate $\int_C (4 + e^{\sqrt{x}})dx + (\sin y + 3x^2)dy$ where C is the boundary of that part of the disk $x^2 + y^2 \leq 16$ in the first quadrant.

18. Let S be a thin spherical shell of radius 2 centered at the origin. Compute the mass of the shell above the plane $z = 1$ where S has mass density function $\delta(x, y, z) = 1 + z^2$

19. Let $\overline{F}(x,y,z) = (2x - y)\overline{i} + 2z\overline{j} + (y - z)\overline{k}$. Find the work W done by the force $\overline{F}$ on an object moving from (0, 0, 0) to (1, 2, 3) along a straight line.

Sample Final Exam — Solution Key

Some other topics not covered on the sample final but required for the final exam:

1. Distance between: point and line
   point and plane
2. Triple scalar product (= volume of box)
3. Vector projections.
4. Tangential & normal components of acceleration
5. Level curves.
6. Finding abs. max. and min. of $f(x,y)$ on closed bounded region.
7. Unoriented line integral (= mass of wire)

~~8. Stokes & Divergence Theorems~~

Solutions: (sorry for random order)

#1 a) Vector normal to P_1: $\bar{n}_1 = \langle 1, -3, 4 \rangle$
-- -- to P_2: $\bar{n}_2 = \langle -1, 2, 0 \rangle$

b) Let $x=0$. Then $y=1$ and from
$0 - 3 \cdot 1 + 4z = 3$, we have $z = 3/2$
$P(0, 1, 3/2)$ is common to P_1 and P_2

c) direction of line $= \bar{v} = \bar{n}_1 \times \bar{n}_2 = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ 1 & -3 & 4 \\ -1 & 2 & 0 \end{vmatrix} =$

$= \langle -8, -4, -1 \rangle$ Line: $\begin{cases} x = -8t \\ y = 1 - 4t \quad -\infty < t < \infty \\ z = 3/2 - t \end{cases}$

#2 $\bar{r} = t\bar{i} - 2\cos t\,\bar{j} - 2\sin t\,\bar{k}$ (helix)

(a) $\bar{v}(t) = \frac{d\bar{r}}{dt} = \bar{i} + 2\sin t\,\bar{j} - 2\cos t\,\bar{k}$

(b) $\bar{a}(t) = \frac{d^2\bar{r}}{dt^2} = 2\cos t\,\bar{j} + 2\sin t\,\bar{k}$

(c) $(\text{speed at } \frac{\pi}{2}) = |\bar{v}(\frac{\pi}{2})| = |\bar{i} + 2\bar{j} - 0\bar{k}| = \sqrt{5}$

(d) $L = \int_0^{\pi} |\bar{v}|\,dt = \int_0^{\pi} \sqrt{1^2 + 4\sin^2 t + 4\cos^2 t}\,dt =$

$= \int_0^{\pi} \sqrt{5}\,dt = \sqrt{5}\,\pi$

(e) $\mathcal{K} = \frac{|\bar{v}\times\bar{a}|}{|\bar{v}|^3}$

$$\bar{v}\times\bar{a} = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ 1 & 2\sin t & -2\cos t \\ 0 & 2\cos t & 2\sin t \end{vmatrix} = 4\bar{i} - 2\sin t\,\bar{j} + 2\cos t\,\bar{k}$$

$|\bar{v}\times\bar{a}| = \sqrt{16 + 4\sin^2 t + 4\cos^2 t} = \sqrt{20} = 2\sqrt{5}$

$\mathcal{K} = \frac{2\sqrt{5}}{(\sqrt{5})^3} = \frac{2}{5}$

#3 $z = 4x^2 - y^2$ is equivalent to $4x^2 - y^2 - z = 0$

$f(x,y,z) = 4x^2 - y^2 - z$

$f_x = 8x$ $\quad f_y = -2y$ $\quad f_z = -1$

$f_x(5,-8,36) = 40$ $\quad f_y(5,-8,36) = 16$ $\quad f_z(5,-8,36) = -1$

$40(x-5) + 16(y+8) - (z-36) = 0$

$\boxed{40x + 16y - z = 36}$

#4 $f(x,y) = 3x^2 - xy$

(a) $df = f_x\,dx + f_y\,dy = (6x-y)\,dx - x\,dy$

(b) $(x_0, y_0) = (1, 2)$

$(x, y) = (1.01, 1.98)$

$f(1.01, 1.98) \approx L(1.01, 1.98) = f(1,2) + df =$

$= (3-2) + (6\cdot 1 - 2)\,0.01 - 1(-0.02) =$

$= 1 + 0.04 + 0.02 = 1.06$

#5. (a) $f(x,y,z) = \sqrt{x^2+y^2+z^2}$

$\nabla f = \frac{x}{\sqrt{x^2+y^2+z^2}}\,\bar{i} + \frac{y}{\sqrt{x^2+y^2+z^2}}\,\bar{j} + \frac{z}{\sqrt{x^2+y^2+z^2}}\,\bar{k}$

$\nabla f(-2,2,1) = -\frac{2}{3}\,\bar{i} + \frac{2}{3}\,\bar{j} + \frac{1}{3}\,\bar{k}$

$\bar{a} = \bar{i} - 2\bar{k}$, $\bar{u} = \frac{\bar{a}}{|\bar{a}|} = \frac{1}{\sqrt{5}}\,i - \frac{2}{\sqrt{5}}\,\bar{k}$

$\left(\frac{df}{ds}\right)_{\bar{u}, P_0} = \nabla f(-2,2,1) \cdot \bar{u} = -\frac{2}{3}\cdot\frac{1}{\sqrt{5}} + \frac{1}{3}\left(-\frac{2}{\sqrt{5}}\right) = -\frac{4}{3\sqrt{5}}$

(b) f increases most rapidly in the direction of $\nabla f(-2,2,1) = -\frac{2}{3}\,\bar{i} + \frac{2}{3}\,\bar{j} + \frac{1}{3}\,\bar{k}$

rate of increase $= |\nabla f(-2,2,1)| = \sqrt{\frac{4}{9} + \frac{4}{9} + \frac{1}{9}} = 1$

#7 $\frac{\partial z}{\partial x} = \frac{\partial z}{\partial r}\cdot\frac{\partial r}{\partial x} + \frac{\partial z}{\partial s}\cdot\frac{\partial s}{\partial x} + \frac{\partial z}{\partial v}\cdot\frac{\partial v}{\partial x} =$

$= 3r^2 e^y + v^2 y e^x + 2sv\cdot 2xy$ (return to x & y

8. Minimize $f(x,y,z) = 4x^2 + y^2 + 5z^2$ subject to $2x + y + 4z = 12$.

Let $g(x,y,z) = 2x + y + 4z$

Solve the system
$$\begin{cases} 2x + y + 4z = 12 \\ 8x = \lambda \cdot 2 \\ 2y = \lambda \cdot 1 \\ 10z = \lambda \cdot 4 \end{cases}$$

$$x = \frac{\lambda}{4}, \quad y = \frac{\lambda}{2}, \quad z = \frac{2}{5}\lambda$$

$$2 \cdot \frac{\lambda}{4} + \frac{\lambda}{2} + 4 \cdot \frac{2}{5}\lambda = 12$$

$$\frac{\lambda}{2} + \frac{\lambda}{2} + \frac{8\lambda}{5} = 12 \quad / \cdot 10$$

$$5\lambda + 5\lambda + 16\lambda = 120$$

$$26\lambda = 120$$

$$\lambda = \frac{120}{26} = \frac{60}{13}$$

f has min. value at $\left(\frac{15}{13}, \frac{30}{13}, \frac{24}{13}\right)$

6

$$\lim_{(x,y)\to(0,0)} f(x,y) = \lim_{x\to 0}\left(x + \frac{2}{x}\right)x = \lim_{x\to 0}\left(x^2+2\right) = 2$$

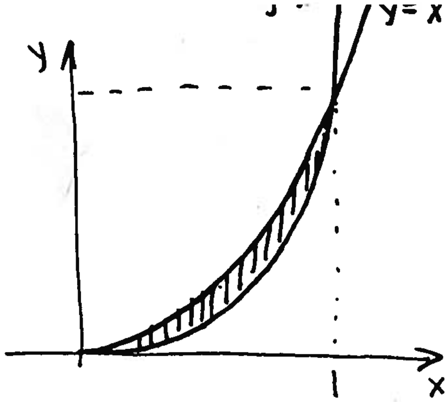
along $y = x$

$$\lim_{(x,y)\to(0,0)} f(x,y) = \lim_{x\to 0}\left(x + \frac{2}{x}\right)\cdot 0 = \lim_{x\to 0} 0 = 0$$

along $y = 0$

$\lim_{(x,y)\to(0,0)} f(x,y)$ does not exist because limits along two different curves are different

11



R: $0 \le x \le 1$
$x^3 \le y \le x^2$

$$\iint_R x^3 y \, dA = \int_0^1 \int_{x^3}^{x^2} x^3 y \, dy \, dx =$$

$$= \int_0^1 \frac{1}{2} x^3 y^2 \Big]_{x^3}^{x^2} dx = \frac{1}{2} \int_0^1 (x^7 - x^9) \, dx =$$

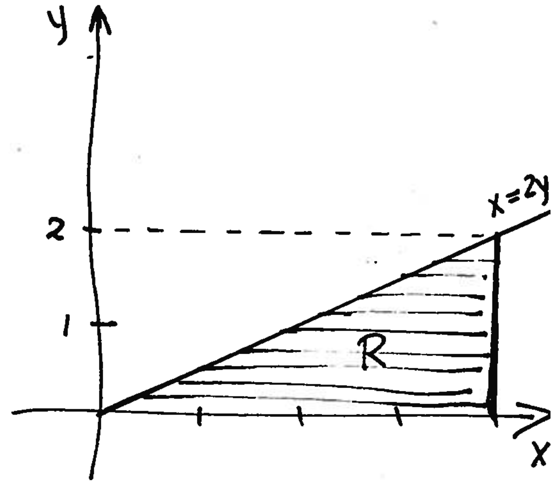
$$= \frac{1}{2} \left(\frac{1}{8} x^8 - \frac{1}{10} x^{10} \right) \Big]_0^1 = \frac{1}{2} \left(\frac{1}{8} - \frac{1}{10} \right) =$$

$$= \frac{1}{2} \left(\frac{5-4}{40} \right) = \frac{1}{80}$$

10

$$I = \int_0^2 \int_{2y}^4 e^{-x^2} dx \, dy$$

R: $0 \le y \le 2$
$2y \le x \le 4$



R: $0 \le x \le 4$
$0 \le y \le \frac{x}{2}$

$$I = \int_0^4 \int_0^{x/2} e^{-x^2} dy \, dx = \int_0^4 e^{-x^2} \cdot \frac{x}{2} \, dx = -\frac{1}{4} \int_0^{-16} e^u \, du =$$

$$= -\frac{1}{4} e^u \Big]_0^{-16} = \frac{1}{4} \left(e^0 - e^{-16} \right) = \frac{1 - e^{-16}}{4}$$

subst.
$u = -x^2$
$du = -2x \, dx$
$-\frac{1}{2} du = x \, dx$
$x = 0 \Rightarrow u = 0$
$x = 4 \Rightarrow u = -16$

9

$f(x,y) = x^3 - 2x^2 - 2xy + y^2$
$f_x = 3x^2 - 4x - 2y = 0$
$f_y = -2x + 2y = 0$ $\rightarrow$ $y = x$,

$3x^2 - 4x - 2x = 0$
$3x^2 - 6x = 0$
$3x(x-2) = 0$
$x = 0$ or $x = 2$

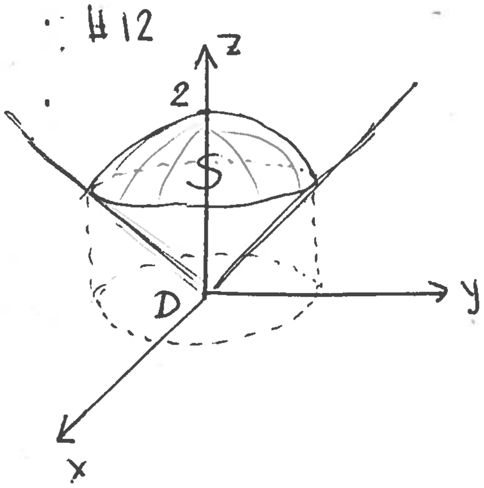
critical points: $(0,0)$, $(2,2)$

$$D(x,y) = \begin{vmatrix} 6x-4 & -2 \\ -2 & 2 \end{vmatrix} = 12x - 8 - 4 = 12x - 12$$

$D(0,0) = -12 \Rightarrow$ $(0,0)$ is a saddle point

$D(2,2) = 12$ and $f_{xx}(2,2) = 8 \Rightarrow$ loc. min. at $(2,2)$, $f(2,2) = -4$

#12



Intersection of the sphere with the cone

$$x^2+y^2+z^2=4$$
$$x^2+y^2=z^2$$

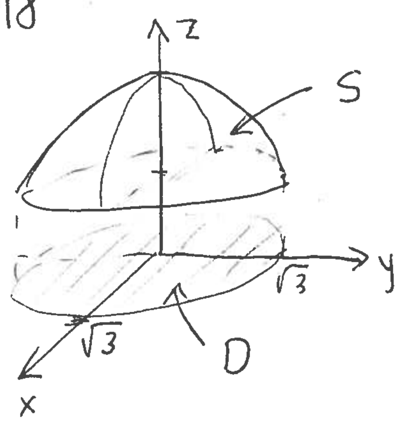
$$2(x^2+y^2)=4$$
$$x^2+y^2=2 \quad \text{- circle of radius } \sqrt{2}$$

$$g(x,y)=\sqrt{4-x^2-y^2}$$
$$g_x=\frac{-x}{\sqrt{4-x^2-y^2}}, \quad g_y=\frac{-y}{\sqrt{4-x^2-y^2}}$$

$$A(S)=\iint_S 1\,dS=\iint_D \sqrt{g_x^2+g_y^2+1}\,dA=\iint_D \frac{2}{\sqrt{4-x^2-y^2}}\,dA$$

D is the disk of radius $\sqrt{2}$ centered at the origin

#18



Intersection of the sphere and the plane $z=1$

$$x^2+y^2+1^2=4$$
$$x^2+y^2=3$$

D: disk of radius $\sqrt{3}$.

$$S: \quad z=g(x,y)=\sqrt{4-x^2-y^2}$$

$$g_x=\frac{-x}{\sqrt{4-x^2-y^2}}, \quad g_y=\frac{-y}{\sqrt{4-x^2-y^2}}, \quad \sqrt{1+g_x^2+g_y^2}=\frac{2}{\sqrt{4-x^2-y^2}}$$

$$\text{Mass}=\iint_S (1+z^2)\,dS=\iint_D (1+4-x^2-y^2)\frac{2}{\sqrt{4-x^2-y^2}}\,dA=$$

polar coord. D: $0\le\theta\le 2\pi$, $0\le r\le\sqrt{3}$

$$=\int_0^{2\pi}\int_0^{\sqrt{3}}(5-r^2)\frac{2}{\sqrt{4-r^2}}\,r\,dr\,d\theta=$$

subst. $u=4-r^2$, $du=-2r\,dr$; $r=0 \Rightarrow u=4$, $r=\sqrt{3}\Rightarrow u=1$

$$=\int_0^{2\pi}\int_1^4 (1+u)\frac{1}{\sqrt{u}}\,du\,d\theta=2\pi\int_1^4 \left(u^{-\frac{1}{2}}+u^{\frac{1}{2}}\right)du=2\pi\left(2u^{1/2}+\frac{2}{3}u^{3/2}\right)\Big|_1^4$$

$$=2\pi\left[4+\frac{16}{3}-2-\frac{2}{3}\right]=2\pi\left[2+\frac{14}{3}\right]=\boxed{\frac{40}{3}\pi}$$

#15 In spherical coordinates G: $0 \le \theta \le 2\pi$
$0 \le \phi \le \frac{\pi}{6}$
$2 \le \rho \le 3$

$$V = \iiint_G 1 \, dV = \int_0^{2\pi} \int_0^{\frac{\pi}{6}} \int_2^3 \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta =$$

$$= \int_0^{2\pi} \int_0^{\pi/6} \left(\frac{1}{3} \rho^3 \sin\phi \right) \Big|_2^3 \, d\phi \, d\theta = \int_0^{2\pi} \int_0^{\frac{\pi}{6}} \frac{19}{3} \sin\phi \, d\phi \, d\theta =$$

$$= \int_0^{2\pi} \left(-\frac{19}{3} \cos\phi \right) \Big|_0^{\frac{\pi}{6}} d\theta = \left(\frac{19}{3} - \frac{19}{3} \cdot \frac{\sqrt{3}}{2} \right) 2\pi = \frac{38\pi}{3} \left(1 - \frac{\sqrt{3}}{2} \right)$$

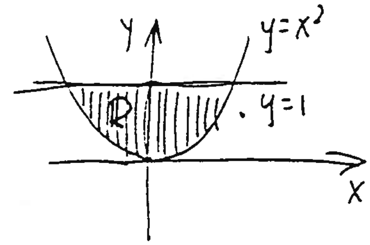
#14 In cylindrical coordinates G: $0 \le \theta \le 2\pi$
$0 \le r \le 2$
$0 \le z \le \sqrt{9 - r^2}$

$$\text{density function} = f(x, y, z) = \sqrt{x^2 + y^2}$$
$$f(r, \theta, z) = r$$

$$\text{mass} = \iiint_G f(x, y, z) \, dV = \int_0^{2\pi} \int_0^2 \int_0^{\sqrt{9-r^2}} r \cdot r \, dz \, dr \, d\theta$$

#13 $z = 1 - y$. If $z = 0$, then $y = 1$

$$V = \iint_R (1 - y) \, dA = \int_{-1}^{1} \int_{x^2}^{1} (1 - y) \, dy \, dx =$$



$$= \int_{-1}^{1} \left(y - \frac{1}{2} y^2 \right) \Big|_{x^2}^{1} dx = \int_{-1}^{1} \left(\frac{1}{2} - x^2 + \frac{1}{2} x^4 \right) dx =$$

$$= \frac{1}{2} x - \frac{1}{3} x^3 + \frac{1}{10} x^5 \Big|_{-1}^{1} = 2 \left(\frac{1}{2} - \frac{1}{3} + \frac{1}{10} \right) = \frac{8}{15}$$

#16 $\frac{\partial P}{\partial y} = 2x - 4y$ $\frac{\partial Q}{\partial x} = 2x - 4y$ $\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$

Finding potential ϕ: $\frac{\partial \phi}{\partial x} = 2xy - 2y^2 \Rightarrow$

$$\phi = \int (2xy - 2y^2)\,dx = x^2y - 2xy^2 + C(y)$$

$$\frac{\partial \phi}{\partial y} = x^2 - 4xy - 4 \Rightarrow x^2 - 4xy + C'(y) = x^2 - 4xy - 4$$

$$C'(y) = -4$$

$$C(y) = -4y + C$$

$$\phi(x,y) = x^2y - 2xy^2 - 4y + C$$

$$\int \bar{F} \cdot d\bar{r} = \phi(-2,3) - \phi(0,1) = (12 + 36 - 12) - (-4) = 40$$

#17 $P(x,y) = 4 + e^{\sqrt{x}}$ $\frac{\partial P}{\partial y} = 0$

$Q(x,y) = \sin y + 3x^2$ $\frac{\partial Q}{\partial x} = 6x$

$$\int_C \bar{F} \cdot d\bar{r} = \iint_D 6x\,dA = \int_0^{\frac{\pi}{2}} \int_0^4 6r\cos\theta\, r\,dr\,d\theta =$$

$$= \int_0^{\frac{\pi}{2}} \left(2r^3\cos\theta\right)\Big|_0^4 d\theta = 128 \int_0^{\frac{\pi}{2}} \cos\theta\,d\theta =$$

$$= 128 \sin\theta \Big|_0^{\frac{\pi}{2}} = 128$$

#19 $W = \int_C \bar{F} \cdot d\bar{r}$

Parametrization of C: $\begin{cases} x = t \\ y = 2t \\ z = 3t \end{cases}$ $0 \le t \le 1$

$$W = \int_0^1 \left[(2t - 2t)1 + 6t \cdot 2 + (2t - 3t)3\right] dt =$$

$$= \int_0^1 9t\,dt = \frac{9}{2}t^2\Big|_0^1 = \frac{9}{2}$$